On Characterization of Time-Like Horizontal Biharmonic Curves in the Lorentzian Heisenberg Group $Heis^3$

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Z. Naturforsch. **65a,** 641 – 648 (2010); received February 12, 2009 / revised September 13, 2009

In this paper, we study energy of time-like horizontal biharmonic curves in the Lorentzian Heisenberg group $Heis^3$. We characterize the biharmonic curves in terms of their curvature and torsion. We prove that all of the biharmonic curves are helices. Finally, we study the mechanics of biharmonic curves and provide conditions for energy of horizontal biharmonic curves.

Key words: Heisenberg Group; Biharmonic Curve; Helices; Horizontal Curve.
Mathematics Subject Classification (2000): 58E20

1. Introduction

The theory of biharmonic functions is an old and rich subject. The biharmonic functions were first studied by Maxwell and Airy to describe a mathematical model of elasticity in 1862. The theory of polyharmonic functions was later on developed, for example, by E. Almansi, T. Levi-Civita, and M. Nicolescu. Recently, biharmonic functions on Riemannian manifolds have been studied by Caddeo and Vanhecke [1, 2], Sario et al. [3], and others.

Chen [4] classified biharmonic curves in semi-Euclidean three-space. In particular, they showed that in Euclidean three-space, there are no proper biharmonic curves (i. e., biharmonic curves which are not harmonic). On the other hand, in indefinite semi-Euclidean three-space, there exist proper biharmonic curves.

Recently, some work has been done in the study of non-geodesic biharmonic curves in some model spaces. For example, the study of biharmonic curves in Berger's spheres, in Cartan-Vranceanu three-dimensional space, in contact and Sasakian manifolds, and in Minkowski three-space, see [5 – 9], respectively.

In the last decade there have been a growing interest in the theory of biharmonic functions which can be divided into two main research directions. On one side, the differential geometric aspect has driven attention to the construction of examples and classification results [10 – 13]. The other side is the analytic aspect from the point of view of a partial differential equation (PDE): biharmonic functions are solutions of a fourth order strongly elliptic semilinear PDE.

Biharmonic functions are utilized in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them. In linear elasticity, if the equations are formulated in terms of displacements for two-dimensional problems, then the introduction of a stress function leads to a fourth-order equation of biharmonic type. For instance, the stress function is proved to be biharmonic for an elastically isotropic crystal undergoing a phase transition, which follows spontaneous dilatation. Biharmonic functions also arise when dealing with transverse displacements of plates and shells. They can describe the deflection of a thin plate subjected to uniform loading over its surface with fixed edges. Biharmonic functions arise in fluid dynamics, particularly in Stokes flow problems, i. e., low-Reynolds-number flows. There are many applications for Stokes flow such as in engineering and biological transport phenomena (for details, see [14, 15]). Fluid flow through a narrow pipe or channel, such as that used in micro-fluidics, involves low Reynolds number. Seepage flow through cracks and pulmonary alveolar blood flow can also be approximated by Stokes flow. Stokes flow also arises in flow through porous media, which have been long applied by civil engineers to groundwater movement. The industrial applications include the fabrication of microelectronic components, the effect of surface roughness on lubrication, the design of polymer dies, and the development of peristaltic pumps for sensitive viscous materials. In natural systems, creeping flows are important in biomedical applications and studies of animal locomotion.

Let $f:(M,g)\to(N,h)$ be a smooth function between two Lorentzian manifolds. The bienergy $E_2(f)$ of f over the compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_\Omega h(\tau(f),\tau(f))\mathrm{d}v_g, \tag{1}$$

where $\tau(f)=\mathrm{trace}_g\nabla\mathrm{d}f$ is the tension field of f and $\mathrm{d}v_g$ is the volume form of M. Using the first variational formula one sees that f is a biharmonic function if and only if its bitension field vanishes identically, i.e.

$$\begin{aligned}\widetilde{\tau}(f):=&-\triangle^f(\tau(f))-\mathrm{trace}_gR^N(\mathrm{d}f,\tau(f))\mathrm{d}f\\ =&\ 0,\end{aligned} \tag{2}$$

where

$$\triangle^f=-\mathrm{trace}_g(\nabla^f)^2=-\mathrm{trace}_g(\nabla^f\nabla^f-\nabla^f_{\nabla^M}) \tag{3}$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN)$ and R^N is the curvature operator of (N,h) defined by

$$R(X,Y)Z=-[\nabla_X,\nabla_Y]Z+\nabla_{[X,Y]}Z,$$

see [16, 17].

Consider the subelliptic operator

$$\Delta_X=\frac{1}{2}(e_2^2+e_3^2),$$

and define the Hamiltonian as the principal symbol of Δ_X

$$H(x,z,\xi,\theta)=\frac{1}{2}\xi_1^2+\frac{1}{2}(\xi_2+x\theta)^2.$$

In the first part we shall consider the dynamics of a charged particle in a constant magnetic field $\Omega=\mathrm{d}\omega$. In other words, we study the dynamics described by the Lagrangian

$$L=\frac{1}{2}g(\gamma',\gamma')+\theta\omega(\gamma').$$

This Lagrangian is the difference between a kinetic energy and the square of a magnetic potential. Hence, we deal with a variational problem with non-holonomic constraints.

In this paper we prove that the biharmonic curves of the Lorentzian Heisenberg group $Heis^3$ are helices, and we find out their explicit parametric equations. In terms of the equations we obtain horizontal biharmonic curves in the Lorentzian Heisenberg group $Heis^3$. We provide conditions for energy of horizontal biharmonic curves. In terms of the curve with the minumum energy between two given points we obtain the energy of the horizontal biharmonic curves. Finally, the relationship between the Hamilton-Jacobi equation and the time-like horizontal biharmonic curve is pointed out in the Lorentzian Heisenberg group $Heis^3$.

2. The Lorentzian Heisenberg Group *Heis*³

The Lorentzian Heisenberg group $Heis^3$ can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\overline{x}y+x\overline{y}).$$

$Heis^3$ is a three-dimensional, connected, simply connected, and two-step nilpotent Lie group.

The Lorentz metric g is given by:

$$g=-\mathrm{d}x^2+\mathrm{d}y^2+(x\mathrm{d}y+\mathrm{d}z)^2,$$

where

$$\omega^1=\mathrm{d}z+x\mathrm{d}y,\quad \omega^2=\mathrm{d}y,\quad \omega^3=\mathrm{d}x$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame

$$e_1=\frac{\partial}{\partial z},\quad e_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\quad e_3=\frac{\partial}{\partial x}, \tag{4}$$

for which we have the Lie products

$$[e_2,e_3]=2e_1,\quad [e_3,e_1]=0,\quad [e_2,e_1]=0,$$

with

$$g(e_1,e_1)=g(e_2,e_2)=1,\quad g(e_3,e_3)=-1, \tag{5}$$

see [18, 19].

Proposition 1: For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g defined above the following is true:

$$\nabla=\begin{pmatrix}0 & e_3 & e_2\\ e_3 & 0 & e_1\\ e_2 & -e_1 & 0\end{pmatrix}, \tag{6}$$

where the (i,j)-element in the table above equals $\nabla_{e_i}e_j$ for our basis

$$\{e_k, k=1,2,3\} = \{e_1,e_2,e_3\}.$$

We adopt the following notation and sign convention for the Riemannian curvature operator:

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Moreover, we put

$$R_{ABC} = R(e_A,e_B)e_C, \quad R_{ABCD} = R(e_A,e_B,e_C,e_D),$$

where the indices A, B, C, and D take the values 1, 2, and 3.

$$R_{121} = -e_2, \quad R_{131} = -e_3, \quad R_{232} = 3e_3$$

and

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{7}$$

3. Biharmonic Curves in the Lorentzian Heisenberg Group $Heis^3$

Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesics time-like curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Let $\{T,N,B\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $Heis^3$ along γ defined as follows:

T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ), and B is chosen so that $\{T,N,B\}$ is a positively oriented orthonormal basis [20]. Then we have the following Frenet formulas:

$$\nabla_T T = \kappa N, \quad \nabla_T N = \kappa T + \tau B, \quad \nabla_T B = -\tau N, \tag{8}$$

where $\kappa = |\tau(\gamma)| = |\nabla_T T|$ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{e_1,e_2,e_3\}$ we can write

$$\begin{aligned} T &= T_1e_1 + T_2e_2 + T_3e_3, \\ N &= N_1e_1 + N_2e_2 + N_3e_3, \\ B &= T \times N = B_1e_1 + B_2e_2 + B_3e_3. \end{aligned}$$

Theorem 2: $\gamma : I \longrightarrow Heis^3$ is a biharmonic curve if and only if

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \quad \kappa^2 - \tau^2 = -1 + 4B_1^2, \\ &\tau' = -2N_1B_1. \end{aligned} \tag{9}$$

Proof. Using (8), we have

$$\begin{aligned} \widetilde{\tau}(\gamma) &= \nabla_T^3 T + \kappa R(T,N)T \\ &= (3\kappa\kappa')T + (\kappa'' + \kappa^3 - \kappa\tau^2)N \\ &\quad + (2\tau\kappa' + \kappa\tau')B + \kappa R(T,N)T. \end{aligned}$$

By (2) we see that γ is a biharmonic curve if and only if

$$\begin{aligned} &\kappa\kappa' = 0, \\ &\kappa'' + \kappa^3 - \kappa\tau^2 = -\kappa R(T,N,T,N), \\ &2\tau\kappa' + \kappa\tau' = -\kappa R(T,N,T,B). \end{aligned} \tag{10}$$

Since $\kappa \neq 0$ by the assumption that is non-geodesic

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^2 - \tau^2 = -R(T,N,T,N), \\ &\tau' = -R(T,N,T,B). \end{aligned} \tag{11}$$

A direct computation using (7) yields

$$\begin{aligned} R(T,N,T,N) &= 1 - 4B_1^2 \\ R(T,N,T,B) &= 2N_1B_1. \end{aligned} \tag{12}$$

These, together with (11), complete the proof of the theorem.

Theorem 3: Let $\gamma : I \longrightarrow Heis^3$ be a time-like curve with constant curvature and $N_1B_1 \neq 0$. Then γ is not biharmonic.

Proof. We can use (6) to compute the covariant derivatives of the vector fields T,N, and B as

$$\begin{aligned} \nabla_T T &= T_1'e_1 + (T_2' + 2T_1T_3)e_2 \\ &\quad + (T_3' + 2T_1T_2)e_3, \\ \nabla_T N &= (N_1' + T_2N_3 - T_3N_2)e_1 \\ &\quad + (N_2' + T_1N_3 - T_3N_1)e_2 \\ &\quad + (N_3' + T_2N_1 - T_1N_2)e_3, \\ \nabla_T B &= (B_1' + T_2B_3 - T_3B_2)e_1 \\ &\quad + (B_2' + T_1B_3 - T_3B_1)e_2 \\ &\quad + (B_3' + T_2B_1 - T_1B_2)e_3. \end{aligned} \tag{13}$$

It follows that the first components of these vectors are given by

$$\begin{aligned}\langle \nabla_T T, e_1\rangle &= T_1',\\ \langle \nabla_T N, e_1\rangle &= N_1' + T_2N_3 - T_3N_2,\\ \langle \nabla_T B, e_1\rangle &= B_1' + T_2B_3 - T_3B_2.\end{aligned} \tag{14}$$

On the other hand, using Frenet formulas (8) we have

$$\begin{aligned}\langle \nabla_T T, e_1\rangle &= \kappa N_1,\\ \langle \nabla_T N, e_1\rangle &= \kappa T_1 + \tau B_1,\\ \langle \nabla_T B, e_1\rangle &= -\tau N_1.\end{aligned} \tag{15}$$

These, together with (14) and (15), give

$$\begin{aligned}T_1' &= \kappa N_1,\\ N_1' + T_2N_3 - T_3N_2 &= \kappa T_1 + \tau B_1,\\ B_1' + T_2B_3 - T_3B_2 &= -\tau N_1.\end{aligned} \tag{16}$$

Assume now that γ is biharmonic. Then using $\tau' = -2N_1B_1 \neq 0$ and (9), we obtain

$$-2\tau\tau' = 8B_1B_1'$$

and

$$\tau N_1B_1 = 2B_1B_1'.$$

Then

$$\tau = \frac{2B_1'}{N_1}. \tag{17}$$

If we use $T_2B_3 - T_3B_2 = N_1$ and (16) we get

$$B_1' = (1-\tau)N_1.$$

We substitute B_1' in equation (17):

$$\tau = \frac{2}{3} = \text{constant}.$$

Therefore also τ is constant and we have a contradiction that is $\tau' = N_1B_1 \neq 0$.

Corollary 4: $\gamma : I \longrightarrow Heis^3$ is biharmonic if and only if

$$\begin{aligned}&\kappa = \text{constant} \neq 0,\\ &\tau = \text{constant},\\ &N_1B_1 = 0,\\ &\kappa^2 - \tau^2 = -1 + 4B_1^2.\end{aligned} \tag{18}$$

Corollary 5: $N_1 \neq 0$ then γ is not biharmonic.

Proof. We use the third equation of (16), we obtain

$$(1-\tau)N_1 = 0. \tag{19}$$

Using $N_1 \neq 0$ we have

$$\tau = 1. \tag{20}$$

Assume now that γ is biharmonic. If we substitue (19) in (9) we obtain

$$\kappa^2 = 4B_1^2. \tag{21}$$

By multipliying both side of (21) with N_1 we obtain

$$\kappa^2 N_1 = 4B_1(B_1N_1).$$

Using (9) and $N_1 \neq 0$ we have $\kappa = 0$. These, together with Theorem 2 complete the proof of the corollary.

Corollary 6: $N_1 = 0$, then

$$\phi = \phi_0 = \text{constant}, \tag{22}$$

where $\phi_0 \in \mathbb{R}$.

Proof. Since γ is s parametrized by arc length, we can write

$$\begin{aligned}T(s) = \sinh\phi(s)e_1 &\\ + \cosh\phi(s)\sinh\psi(s)e_2 &\\ + \cosh\phi(s)\cosh\psi(s)e_3.&\end{aligned} \tag{23}$$

From (16) we obtain

$$T_1' = \kappa N_1.$$

Since $N_1 = 0$ we have

$$T_1' = 0.$$

Then T_1 is constant. Using (23) we get

$$T_1 = \sinh\phi_0 = \text{constant}.$$

We obtain (22) and the corollary is proved.

Theorem 7: The parametric equations of all time-like biharmonic curves are:

$$\begin{aligned}x(s) &= \frac{1}{\digamma}\cosh\phi_0\sinh(\digamma s + \rho) + c_1,\\ y(s) &= \frac{1}{\digamma}\cosh\phi_0\cosh(\digamma s + \rho) + c_2,\\ z(s) &= \sinh\phi_0 s - \frac{1}{2\digamma^2}[\cosh\phi_0]^2\sinh 2(\digamma s + \rho)\\ &\quad - \frac{[\cosh\phi_0]^2}{\digamma}s - \frac{c_1}{\digamma}\cosh\phi_0\cosh(\digamma s + \rho) + c_3,\end{aligned} \tag{24}$$

where $\mathcal{F} = \left(\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0\right)$ and $\phi_0, c_1, c_2, c_3, \rho \in \mathbb{R}$.

Proof. The covariant derivative of the vector field T is

$$\nabla_T T = T_1' e_1 + (T_2' + 2T_1T_3)e_2 + (T_3' + 2T_1T_2)e_3.$$

From (23) we have

$$\begin{aligned}&\nabla_T T = \\ &(\psi' \cosh\phi_0 \cosh\psi(s) + 2\sinh\phi_0 \cosh\phi_0 \cosh\psi(s))e_2 \\ &+ (\psi' \cosh\phi \sinh\psi(s) + 2\sinh\phi_0 \cosh\phi_0 \cosh\psi(s))e_3.\end{aligned}$$

Since $|\nabla_T T| = \kappa$ we obtain

$$\psi(s) = \left(\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0\right)s + \rho, \tag{25}$$

where $\rho \in \mathbb{R}$.

To find equations for the time-like biharmonic curve $\gamma(s) = (x(s), y(s), z(s))$ on the Lorentzian Heisenberg group *Heis*^{3} we note that if

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = T = T_1e_1 + T_2e_2 + T_3e_3$$

and our left-invariant vector fields are

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x},$$

then

$$\frac{\partial}{\partial x} = e_3, \quad \frac{\partial}{\partial y} = e_2 + xe_3, \quad \frac{\partial}{\partial z} = e_1.$$

Therefore we easily have

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}s} &= \cosh\phi_0 \cosh\left[\left(\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0\right)s + \rho\right], \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= \cosh\phi_0 \sinh\left[\left(\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0\right)s + \rho\right], \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= \cosh\phi_0 \cosh\left[\left(\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0\right)s + \rho\right] \\ &\quad - x(s)\cosh\phi_0 \sinh\left[\left(\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0\right)s + \rho\right].\end{aligned} \tag{26}$$

If the system (26) is integrated, we obtain (24) and the theorem is proved.

4. Horizontal Biharmonic Curves in the Lorentzian Heisenberg Group *Heis*^{3}

Consider a non-integrable two-dimensional distribution $(x,y) \longrightarrow \mathcal{H}_{(x,y)}$ in $\mathbb{R}^3 = \mathbb{R}^2_{(x,y)} \times \mathbb{R}_z$ defined as $\mathcal{H} = \ker\omega$, where ω is a one-form on $\mathbb{R}^3$. The distribution $\mathcal{H}$ is called the horizontal distribution.

A curve $s \longrightarrow \gamma(s) = (x(s), y(s), z(s))$ is called horizontal curve if $\gamma'(s) \in \mathcal{H}_{\gamma(s)}$, for every s. As

$$\begin{aligned}\gamma'(s) &= x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z \\ &= x'(s)e_3 + y'(s)e_2 + \omega(\gamma'(s))\partial_z,\end{aligned}$$

then $\gamma(s)$ is a horizontal curve if

$$\begin{aligned}&\gamma'(s) = x'(s)e_3 + y'(s)e_2, \\ &\omega(\gamma'(s)) = z'(s) + x(s)y'(s).\end{aligned} \tag{27}$$

If $\gamma(s)$ is a horizontal curve, then we have

$$\begin{aligned}\gamma'(s) &= x'(s)e_3 + y'(s)e_2 \\ &= x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} - x(s)y'(s)\frac{\partial}{\partial z}.\end{aligned} \tag{28}$$

Using (4) and (28) we obtain

$$T = T_3\frac{\partial}{\partial x} + T_2\frac{\partial}{\partial y} + (T_1 - x(s)T_2)\frac{\partial}{\partial z}. \tag{29}$$

Theorem 8: The parametric equations of all time-like horizontal biharmonic curves are

$$\begin{aligned}x(s) &= \frac{1}{\mathcal{F}}\sinh(\mathcal{F}s + \rho) + a_1, \\ y(s) &= \frac{1}{\mathcal{F}}\cosh(\mathcal{F}s + \rho) + a_2, \\ z(s) &= \frac{1}{4\mathcal{F}}\sinh 2(\mathcal{F}s + \rho) - \frac{a_1}{\mathcal{F}}\cosh(\mathcal{F}s + \rho) \\ &\quad - \frac{1}{\mathcal{F}}s + a_3,\end{aligned} \tag{30}$$

where $\mathcal{F} = \left(\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0\right)$ and $\phi_0, a_1, a_2, a_3, \rho \in \mathbb{R}$.

Proof. Using (28) and (29) we have

$$T_1 = \sinh\phi_0 = 0. \tag{31}$$

Substituting (31) into (24) we get (30).

A plot of the time-like horizontal biharmonic curve $\gamma(s)$ at $\mathcal{F} = a_1 = a_2 = a_3 = \rho = 1$ shows Figure 1.

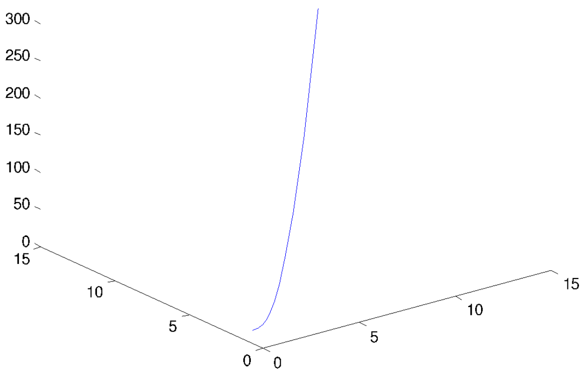

Fig. 1. Time-like horizontal biharmnic curve $\gamma(s)$ at $\digamma = a_1 = a_2 = a_3 = \rho = 1$.

5. Energy of Horizontal Biharmonic Curves in *Heis*^{3}

The Lorentzian Heisenberg group $Heis^3$ is a good environment to apply the Hamiltonian formalism. Consider the Hamiltonian $H : T^*\mathbb{R}^3_{(x,z)} \longrightarrow \mathbb{R}$ given by

$$H(x,z,\xi,\theta) = \frac{1}{2}\xi_1^2 + \frac{1}{2}(\xi_2 + x\theta)^2,$$

which is the principal symbol of the Heisenberg Laplacian

$$\Delta_H = \frac{1}{2}(e_2^2 + e_3^2),$$

where

$$e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}.$$

Also the Hamiltonian is defined as

$$H(x,p) = \frac{1}{2}\sum_k p(e_k)^2.$$

If $p = \mathrm{d}\gamma$

$$H(x,\mathrm{d}\gamma) = \frac{1}{2}\sum_k \mathrm{d}\gamma(e_k)^2 = \frac{1}{2}\sum_k e_k(\gamma)^2 = \frac{1}{2}|\nabla_X(\gamma)|^2.$$

In quantum mechanics, the procedure of obtaining the operator Δ_H from the Hamiltonian $H(x,z,\xi,\theta)$ is called quantization.

It is natural to consider the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial \xi}, \quad \dot{z} = \frac{\partial H}{\partial \theta}, \quad \dot{\xi} = -\frac{\partial H}{\partial x}, \quad \dot{\theta} = -\frac{\partial H}{\partial t}.$$

The solutions $S = (x,z,\xi,\theta)$ of this system are called bicharacteristics.

Lemma 9: Let $\gamma : I \longrightarrow Heis^3$ be a time-like horizontal curve. Then

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial f}{\partial s} + g(\gamma', \nabla_X f), \tag{32}$$

where $X = e_2 + e_3$ and $f \in \mathcal{F}(\mathbb{R}^3)$.

Proof.

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}s} &= \frac{\partial f}{\partial s} + \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial x_2}y' + \frac{\partial f}{\partial z}z', \\ &= \frac{\partial f}{\partial s} + x'e_3(f) + y'e_2(f) + (xy' + z')e_1(f), \\ &= \frac{\partial f}{\partial s} + x'e_3(f) + y'e_2(f) + \omega(\gamma')e_1(f).\end{aligned}$$

Since γ is horizontal curve we have (32).

Theorem 10: Let $\gamma : I \longrightarrow Heis^3$ be a time-like horizontal curve. Then γ is energy-minimizing if and only if

$$\gamma' = \nabla_X S, \tag{33}$$

where $S \in \mathcal{F}$ is the solution for the Hamilton-Jacobi equation and $X = e_2 + e_3$.

Proof. The curve with the minumum energy between two given points

$$E = \int_0^\tau \frac{1}{2}|\gamma'(s)|\mathrm{d}s. \tag{34}$$

Let $S \in \mathcal{F}$ is the solution for the Hamilton-Jacobi equation.

$$\frac{\partial S}{\partial \tau} + \frac{1}{2}|\nabla_X S|^2 = 0, \quad S(O) = 0. \tag{35}$$

Consider the integral

$$\sigma = \int_0^\tau \frac{1}{2}|\gamma'(s)|\mathrm{d}s - \mathrm{d}S. \tag{36}$$

Using Lemma 9 we have

$$\begin{aligned}\sigma &= \int_0^\tau \frac{1}{2}|\gamma'(s)|\mathrm{d}s - \mathrm{d}S \\ &= \int_0^\tau \left(\frac{1}{2}|\gamma'(s)|\mathrm{d}s - \frac{\partial S}{\partial s} - g(\gamma', \nabla_X S)\right)\mathrm{d}s \\ &= \int_0^\tau \left(\frac{1}{2}|\gamma'(s) - \nabla_X S|^2 - \left(\frac{\partial S}{\partial s} + \frac{1}{2}|\nabla_X S|^2\right)\right)\mathrm{d}s.\end{aligned}$$

From (35) we obtain

$$\sigma = \frac{1}{2}\int_0^{\tau} |\gamma'(s) - \nabla_X S|^2 \mathrm{d}s.$$

The energy E and σ reach the minimum for the same horizontal curve γ. This proves the claim.

Corollary 11: Let $\gamma : I \longrightarrow Heis^3$ be a time-like horizontal biharmonic curve. Then,

$$|\nabla_X S|^2 = -1, \tag{37}$$

where $S \in \mathcal{F}$ is the solution for the Hamilton-Jacobi equation.

Proof. From (27) and (33) we obtain

$$\nabla_X S = x'(s)e_3 + y'(s)e_2. \tag{38}$$

Since (5), (38) is rewritten as

$$|\nabla_X S|^2 = -(x'(s))^2 + (y'(s))^2. \tag{39}$$

From Theorem 8 we obtain

$$x'(s) = \cosh(Fs+\rho), \quad y'(s) = \sinh(Fs+\rho). \tag{40}$$

Substituting (40) into (39) we get (37).

Corollary 12: Let $\gamma : I \longrightarrow Heis^3$ be a time-like horizontal biharmonic curve. Then,

$$S = -\frac{1}{2}s + c, \tag{41}$$

where $c \in \mathbb{R}$ and $S \in \mathcal{F}$ is the solution for the Hamilton-Jacobi equation.

Proof. By applying Lemma 9 we get

$$\frac{\partial S}{\partial s} = \frac{\mathrm{d}S}{\mathrm{d}s} - g(\gamma'(s), \nabla_X S).$$

From (35) we obtain

$$-\frac{1}{2}g(\gamma'(s), \gamma'(s)) = \frac{\mathrm{d}S}{\mathrm{d}s} - g(\gamma'(s), \nabla_X S). \tag{42}$$

Also, substituting (33) into (21) gives us

$$\frac{\mathrm{d}S}{\mathrm{d}s} = \frac{1}{2}g(\gamma'(s), \gamma'(s)). \tag{43}$$

Hence, integrating (43), we get (41). The proof is finished.

Example 13 *(Physical relevance of specific problem):*

The front wave is given by the level curves of the energy and it is given by the equation

$$\frac{1}{2}((e_2\gamma)^2 + (e_3\gamma)^2) = k, \tag{44}$$

where k is positive constant.

Consider the energy associated to a time-like horizontal biharmonic curve $\gamma \in \mathcal{F}(\mathbb{R}^3)$. Using the Hamiltonian with γ we have

$$H(\nabla\gamma) = H(x, \mathrm{d}\gamma) = \frac{1}{2}|\nabla_X(\gamma)|^2 = \frac{1}{2}((e_2\gamma)^2 + (e_3\gamma)^2).$$

These, together with level curves of the energy we have (44).

Acknowledgement

The authors thank the referees for their suggestions and advice.

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